M-math (April,2019) Back paper Exam
Subject : Stochastic Processes

Time : 3.00 hours Max.Marks 50.

1. Let $(X_t, t \geq 0)$ have the same finite dimensional distributions as a one dimensional Brownian motion. Show that (X_t) has a modification whose paths are almost surely *Hölder* continuous of order γ for every $\gamma, 0 < \gamma < \frac{1}{2}$. (10)

2. Let $T = \{1, 2, \cdots\}$ and $\{\mu_i, i \in T\}$ a family of probability measures on $\mathbb{R}$. For $t_i \in T, i = 1, \cdots, k$ define probability measures $\mu_{t_1,\cdots,t_k}$ on $\mathbb{R}^k$ as follows: $\mu_{t_1,\cdots,t_k} := \otimes_{j=1}^k \mu_{t_j}$. Show that the family $\{\mu_{t_1,\cdots,t_k}, t_i \in T, 1 \leq i \leq k, k \geq 1\}$ is a consistent family of probability measures. (10)

3. Let (W_t) be standard d-dimensional Brownian motion with filtration $(\mathcal{F}_t^W)$. Show that the strong Markov property holds at a finite stopping time τ iff $\mathcal{F}_\tau^W$ is independent of $(W_{t+\tau} - W_\tau, t \geq 0)$ and the latter process is a standard d-dimensional Brownian motion. (10)

4. Let (W_t) be a continuous real valued process, with $W_0 = 0$, almost surely. Show that (W_t) is a standard Brownian motion iff it is a Gaussian process with mean $EW_t = 0, t \geq 0$ and covariance $Cov(W_t, W_s) = t \wedge s, 0 \leq s, t < \infty$. (10)

5. Let (W_t) be a standard one dimensional Brownian motion and for $\alpha > 0$, let $\tau_\alpha := \inf\{t > 0 : W_t \geq \alpha\}$. Show that τ_α has the density

$$\phi_\alpha(t) := \frac{\alpha}{\sqrt{2\pi}} \frac{1}{t^{\frac{3}{2}}} e^{\frac{-\alpha^2}{2t}}, \quad 0 < t < \infty.$$

(10)